Cubic polynomials

1. Given a cubic polynomial, $p(x)$. If $p(1) = -6$, $p(2) = -4$, $p(3) = 2$ and $p(4) = 18$, find the value of $p(0)$.

1. Method 1

Let $p(x) = Ax^3 + Bx^2 + Cx + D$.

$$p(1) = A + B + C + D = -6 \quad \text{....}(1)$$
$$p(2) = 8A + 4B + 2C + D = -4 \quad \text{....}(2)$$
$$p(3) = 27A + 9B + 3C + D = 2 \quad \text{....}(3)$$
$$p(4) = 256A + 64B + 4C + D = 18 \quad \text{....}(4)$$

$(2) - (1)$, $\quad 7A + 3B + C = 2 \quad \text{....}(4)$

$(3) - (2)$, $\quad 19A + 5B + C = 6 \quad \text{....}(5)$

$(4) - (3)$, $\quad 229A + 55B + C = 16 \quad \text{....}(6)$

$(5) - (4)$, $\quad 12A + 2B = 4$

$\quad 6A + B = 2 \quad \text{....}(7)$

$(3) - (2)$, $\quad 210A + 50B = 10$

$\quad 21A + 5B = 1 \quad \text{....}(8)$

$(7) \times 5$, $\quad 30A + 5B = 10 \quad \text{....}(9)$

$(9) - (8)$, $\quad 9A = 9$

$\therefore \quad A = 1 \quad \text{....}(10)$

$(10) \downarrow (8)$, $\quad B = -4 \quad \text{....}(11)$

$(10), (11) \downarrow (4)$, $\quad C = 7 \quad \text{....}(12)$

$(10), (11), (12) \downarrow (4)$, $\quad D = -10 \quad \text{....}(13)$

$\therefore \quad p(x) = x^3 - 4x^2 + 7x - 10$

$\therefore \quad p(0) = \mathbf{-10}$

Method 2

Let $p(x) = A(x-1)(x-2)(x-3) + B(x-1)(x-2) + C(x-1) + D$.

$$p(1) = D = -6 \quad \text{....}(1)$$
$$p(2) = C + D = -4 \quad \text{....}(2)$$
$$p(3) = 2B + 2C + D = 2 \quad \text{....}(3)$$
$$p(4) = 6A + 6B + 3C + D = 18 \quad \text{....}(4)$$

$(1) \downarrow (2)$, $\quad C = 2 \quad \text{....}(5)$

$(1), (5) \downarrow (3)$, $\quad B = 2 \quad \text{....}(6)$

$(1), (5), (6) \downarrow (4)$, $\quad A = 1 \quad \text{....}(7)$

$$p(x) = (x-1)(x-2)(x-3) + 2(x-1)(x-2) + 2(x-1) - 6$$

$\therefore \quad p(0) = (-1)(-2)(-3) + 2(-1)(-2) + 2(-1) - 6 = \mathbf{-10}$

Method 3

Let

$$p(x) = \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)}(-6) + \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)}(-4) + \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)}(2) + \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}(18)$$

Note that $p(1) = -6$, $p(2) = -4$, $p(3) = 2$ and $p(4) = 18$

$\therefore \quad p(0) = \frac{(-2)(-3)(-4)}{(1-2)(1-3)(1-4)}(-6) + \frac{(-1)(-3)(-4)}{(2-1)(2-3)(2-4)}(-4) + \frac{(-1)(-2)(-4)}{(3-1)(3-2)(3-4)}(2) + \frac{(-1)(-2)(-3)}{(4-1)(4-2)(4-3)}(18)$

$= \mathbf{-10}$

2. Find a cubic polynomial whose graph has horizontal tangents at $(-2, 12)$ and $(4, -6)$.

2. Method 1

Let the polynomial be $p(x) = ax^3 + bx^2 + cx + d$

Since $(-2, 12)$ and $(4, -6)$ are on $p(x)$, $p(-2) = 12$, $p(4) = -6$

Hence, $-8a + 4b - 2c + d = 12 \quad \ldots.(1)$

$64a + 16b + 4c + d = -6 \quad \ldots.(2)$

Differentiate $p(x)$, $p'(x) = 3ax^2 + 2bx + c$

Since $p(x)$ has horizontal tangents at $(-2, 12)$ and $(4, -6)$, $p'(-2) = 0$, $p'(4) = 0$

$12a - 4b + c = 0 \quad \ldots.(3)$

$48a + 8b + c = 0 \quad \ldots.(4)$

$(2) - (1)$, $72a + 12b + 6c = -18 \Longrightarrow 12a + 2b + c = -3 \quad \ldots.(5)$

$(5) - (3)$, $\quad 6b = -3 \Longrightarrow b = -\frac{1}{2} \quad \ldots.(6)$

$(4) - (3)$, $\quad 36a + 12b = 0 \Longrightarrow 3a + b = 0 \Longrightarrow 3a - \frac{1}{2} = 0 \Longrightarrow a = \frac{1}{6} \quad \ldots.(7)$

$(6), (7) \downarrow (3)$, $\quad 12\left(\frac{1}{6}\right) - 4\left(-\frac{1}{2}\right) + c = 0 \Longrightarrow c = -4 \quad \ldots.(8)$

$(6), (7), (8) \downarrow (1)$, $\quad -8\left(\frac{1}{6}\right) + 4\left(-\frac{1}{2}\right) - 2(-4) + d = 12 \Longrightarrow d = \frac{22}{3}$

$\therefore$ The polynomial is $p(x) = \mathbf{\frac{x^3}{6} - \frac{x^2}{2} - 4\,x + \frac{22}{3}}$

Method 2

Let the polynomial be $p(x) = a(x+2)^3 + b(x+2)^2 + c(x+2) + 12$

Note that $(-2, 12)$ is on $p(x)$ and $p(-2) = 12$

Now $(4, -6)$ is on $p(x)$, $p(4) = -6$

$$a(4+2)^3 + b(4+2)^2 + c(4+2) + 12 = -6 \Rightarrow 216a + 36b + 6c = -18 \quad \ldots.(1)$$

Differentiate $p(x)$, $p'(x) = 3a(x+2)^2 + 2b(x+2) + c$

Since $p(x)$ has horizontal tangents at $(-2, 12)$ and $(4, -6)$, $p'(-2) = 0$, $p'(4) = 0$

Therefore, $c = 0 \quad \ldots.(2)$

and $3a(4+2)^2 + 2b(4+2) + c = 0 \Rightarrow 3a\,(4+2)^2 + 2b(4+2) = 0$

$$\Rightarrow 9a + b = 0 \quad \ldots.(3)$$

$(2) \downarrow (1)$, $216a + 36b = -18 \Rightarrow 12a + 2b = -1 \quad \ldots.(4)$

Solving (3) and (4), $a = \frac{1}{6}$ and $b = -\frac{3}{2}$

$\therefore$ The polynomial is $p(x) = \frac{1}{6}(x+2)^3 - \frac{3}{2}(x+2)^2 + 12 = \mathbf{\frac{x^3}{6} - \frac{x^2}{2} - 4\,x + \frac{22}{3}}$

Method 3

Let the polynomial be $p(x) = a(x-4)^2(x+2) + b(x-4)(x+2) + c(x+2) + 12$

Note that $(-2, 12)$ is on $p(x)$ and $p(-2) = 12$

Now $(4, -6)$ is on $p(x)$, $p(4) = -6$

$$c(4+2) + 12 = -6 \Rightarrow 6c = -18 \Rightarrow c = -3 \quad \ldots.(1)$$

Differentiate $p(x)$, $p'(x) = a(x-4)^2 + 2a(x-4)(x+2) + b(x+2) + b(x-4) + c$

Since $p(x)$ has horizontal tangents at $(-2, 12)$ and $(4, -6)$, $p'(-2) = 0$, $p'(4) = 0$

Therefore, $36a - 6b + c = 0 \Rightarrow 36a - 6b - 3 = 0 \Rightarrow 12a - 2b = 1 \quad \ldots.(2)$

and $6b + c = 0 \Rightarrow 6b - 3 = 0 \Rightarrow b = \frac{1}{2} \quad \ldots.(3)$

$(3) \downarrow (2)$, $12a - 1 = 1 \Rightarrow a = \frac{1}{6}$

$\therefore$ The polynomial is $p(x) = \frac{1}{6}(x-4)^2(x+2) + \frac{1}{2}(x-4)(x+2) - 3(x+2) + 12$

$$= \mathbf{\frac{x^3}{6} - \frac{x^2}{2} - 4\,x + \frac{22}{3}}$$

Method 4

Let the polynomial be $p(x)$.

Since $p(x)$ has horizontal tangents at $(-2, 12)$ and $(4, -6)$ and $p(x)$ has turning points there, $p'(x) = k(x+2)(x-4) = k(x^2 - 2x - 8)$

$$p(x) = \int k(x^2 - 2x - 8)dx = \frac{kx^3}{3} - kx^2 - 8k\,x + C$$

Since $(-2, 12)$ and $(4, -6)$ are on $p(x)$, $p(-2) = 12$, $p(4) = -6$

Hence, $\frac{k(-2)^3}{3} - k(-2)^2 - 8\,k(-2) + C = 12 \Rightarrow \frac{28\,k}{3} + C = 12 \quad \ldots.(1)$

$$\frac{k(4)^3}{3} - k(4)^2 - 8\,k(4) + C = -6 \Rightarrow -\frac{80\,k}{3} + C = -6 \quad \ldots.(2)$$

$(1) - (2)$, $36k = 18 \Rightarrow k = \frac{1}{2} \quad \ldots.(3)$

$(3) \downarrow (1)$, $\frac{28}{3} \times \frac{1}{2} + C = 12 \Rightarrow C = \frac{22}{3}$

$\therefore$ The polynomial is $p(x) = \mathbf{\frac{x^3}{6} - \frac{x^2}{2} - 4\,x + \frac{22}{3}}$

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